May not sum to total due to rounding: the probability of rounding errors

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Abstract

Many published datasets use rounded data, either because the measurement process has limited precision, or to improve presentation in the publication. This introduces an error of up to half the rounding precision, but when several pieces of rounded data are added together, the error in the sum can be larger. If the actual sum is also known, but presented as rounded to the same level, then this can be visible as the rounded components may not add up to the rounded sum; it can also be clear when the components are presented as rounded percentages but these do not add up 100%. Where this happens, publishers often notate tables with the warning *May not sum to total due to rounding* or something similar.

This note explores some of the probabilities of rounding errors and visible rounding differences in sums, and provides tables of probabilities of different rounding errors. If many of the individual pieces of data are as small as the rounding precision or smaller, then this may also introduce bias into the sum of the rounding errors: three theoretical examples are considered, together with some actual data showing visible rounding differences.

1 Cumulative rounding errors

Initially we shall assume that the rounding error on individual components, the difference between the rounded and unrounded figure, is independently, continuously and uniformly distributed between $-\frac{1}{2}$ and $\frac{1}{2}$ times the rounding precision, with zero probability of being at the extremes. We can then find the probability of distribution of the error of the sum by convoluting uniform distributions. Unfortunately this soon becomes complicated, though perfectly manageable using integer arithmetic with an unlimited precision computer. Writing $F_n(x) = \text{Prob}(\sum_{i=1}^{n} X_i \leq x)$ for the cumulative distribution when n independent rounding errors are combined, we get for small n

$$F_1(x) = \frac{2x+1}{2} \quad \text{when} \quad -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}.$$

$$F_2(x) = \begin{cases} \frac{x^2+2x+1}{2} & \text{when} \quad -1 \leq x \leq 0 \\ \frac{-x^2+2x+1}{2} & \text{when} \quad 0 \leq x \leq 1. \end{cases}$$

$$F_3(x) = \begin{cases} \frac{8x^3+36x^2+54x+27}{48} & \text{when} \quad -\frac{3}{2} \leq x \leq -\frac{1}{2} \\ \frac{-4x^3+9x+6}{12} & \text{when} \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{8x^3-36x^2+54x+21}{48} & \text{when} \quad \frac{1}{2} \leq x \leq \frac{3}{2}. \end{cases}$$

$$F_4(x) = \begin{cases} \frac{x^4+8x^3+24x^2+36x+16}{24} & \text{when} \quad -2 \leq x \leq -1 \\ \frac{-3x^4-8x^3+16x+12}{24} & \text{when} \quad -1 \leq x \leq 0 \\ \frac{3x^4-8x^3+16x+12}{24} & \text{when} \quad 0 \leq x \leq 1 \\ \frac{-x^4+8x^3-24x^2+36x+8}{24} & \text{when} \quad 1 \leq x \leq 2. \end{cases}$$

$$F_5(x) = \begin{cases} \frac{32x^5+400x^4+2000x^3+5000x^2+6250x+3125}{3840} & \text{when} \quad -\frac{5}{2} \leq x \leq -\frac{3}{2} \\ \frac{-64x^5-400x^4-800x^3-200x^2+1100x+955}{1920} & \text{when} \quad -\frac{3}{2} \leq x \leq -\frac{1}{2} \\ \frac{48x^5-200x^3+575x+480}{960} & \text{when} \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{-64x^5+400x^4-800x^3+200x^2+1100x+965}{1920} & \text{when} \quad \frac{1}{2} \leq x \leq \frac{3}{2} \\ \frac{32x^5-400x^4+2000x^3-5000x^2+6250x+715}{3840} & \text{when} \quad \frac{3}{2} \leq x \leq \frac{5}{2}. \end{cases}$$

Some of these can be factorised or otherwise simplified, but this becomes more difficult as n increases. An alternative is to use the central limit theorem, noting that the initial uniform distribution has mean 0 and variance $\frac{1}{12}$, and so (assuming independence) the sum of n such random variables has mean 0 and variance $\frac{n}{12}$; for large n and x, this indeed provides reasonable approximations, though they are proportionately poor in the tails.

So $F_n(x)$ is the probability that with n rounded numbers, the error in the sum due to rounding is less than or equal to x times the rounding precision. Clearly

$$F_n(x) = 0 \text{ when } x \leq -\frac{n}{2}$$

$$F_n(x) = 1 \text{ when } x \geq \frac{n}{2}$$

and the extreme events $\sum_{i=1}^{n} X_i = \pm\frac{n}{2}$ are possible but of zero probability.

In $F_n(x)$, x is signed; more often, we may be interested in whether the absolute value of the error in the sum is less than or equal to x times the rounding precision for non-negative x. Calling this $G_n(x)$, we have

$$G_n(x) = F_n(x) - F_n(-x) = 2F_n(x) - 1.$$

and so $G_n(x) = 1$ when $x \geq \frac{n}{2}$.

Rounding just one number gives a rounding error that is less than or equal to half the rounding precision. The probability this remains the case with the sum of n rounded numbers is $G_n(\frac{1}{2})$, while the probability that the error in the sum is less than the rounding precision is $G_n(1)$.

Snedecor and Cochran[1] provide an example problem along these lines:

> *Example 4.8.8. When measurements are rounded to the nearest whole number, it can often be assumed that the error due to rounding is equally likely to lie anywhere between* -0.5 *and* $+0.5$*. That is, rounding errors follow a uniform distribution between the limits* -0.5 *and* $+0.5$*. From theory, this distribution has* $\mu = 0$, $\sigma = 1/\sqrt{12} = 0.29$*. If* 100 *independent, rounded measurements are added, what is the probability that the error in the total due to rounding does not exceed* 5 *in absolute value? Ans.* $P = 0.916$.

Clearly the problem is asking us to find $G_{100}(5)$. Calculating the hundred convolutions gives the precise probability of

89 120254 737023 412678 295281 505362 351118 315198 761494 175906 148647 751101 687378 532185 619182 956244 927133 663945 989014 866250 791030 484901 069916 977940 250347 876393 / 97 214807 754108 492376 770040 475277 813011 162466 942064 189029 784337 390851 666659 614495 425980 691641 829706 548180 935334 195024 748178 317928 038400 000000 000000 000000, about 0.91674.

The hint about the value of σ and the position in the book suggest that problem does not expect such precision and instead expects use of the central limit theorem as a reasonable approximation. For a Gaussian distribution, the probability of being within an interval 2.9 standard deviations either side of the mean is about 0.91532, while being more precise about the square root of $\frac{1}{12}$ would give a probability of about 0.91674, suggesting that the provided answer faces some rounding issues of its own.

2 Visible rounding differences

That kind of analysis gives some indication of the probable sizes of errors introduced to a sum as a result of rounding. However, the actual error is not generally visible to external observers. But when individual items and the total are each rounded after the calculation of the unrounded sum, it is sometimes possible to observe that the rounded total is not equal to the sum of the rounded items. To remind readers of this and to avoid unnecessary enquires, many tables include the caveat *May not sum to total due to rounding* or something similar.

The analysis of the possible difference between sum of the rounded parts and the separately rounded total is similar to that before, but this time the rounded total must be an integer multiple of the rounding precision. So the probability that no error is visible is $F_n(\frac{1}{2}) - F_n(-\frac{1}{2}) = G_n(\frac{1}{2})$. If the Snedecor and Cochran example had been reworded to ask *What is the probability that the difference between the sum of the rounded individual measurements and the rounded total does not exceed* 5 *in absolute value?* then the answer would be $G_{100}(5\frac{1}{2})$ or about 0.94325. We could go further and look at the probabilities of particular values of the signed difference between the sum of n rounded parts and the rounded total: if this difference was d times the rounding precision and the probability was $H_n(d)$ then

$$H_n(d) = F_n(d + \tfrac{1}{2}) - F_n(d - \tfrac{1}{2}).$$

3 Visible percentage differences

In many tables of sums, values themselves are not shown, and instead the parts are shown as percentages of the total at 100. Depending on how many places are shown, this may involve rounding with more or less precision. Clearly the independence assumption used before has been lost, and the results will not be the same. To illustrate this, if there are only two parts, they can produce a visible rounding difference in the sum, but they cannot as percentages except in extreme cases: for example, 32.343 and 44.234 summing to 76.577 produces a difference when rounded but 42.236% and 57.764% summing to 100% does not. 42.5% and 57.5% might produce a visible rounding error depending on the rounding method, but this is by assumption a zero probability event.

Fortunately, it is possible to handle find a reasonable first approximation to using rounded percentages despite losing independence. If we only look at the fractional part of the rounding error for one part, our assumption is that this is uniform on $[-\frac{1}{2}, \frac{1}{2}]$ times the rounding precision, as shown by the derivative of $F_1(x)$. But for the fractional part it is also uniform on the same interval (modulo 1) for two parts, and by induction it is therefore also uniform on the same interval (modulo 1) for $n-1$ parts.

So if a further part is needed to reach the unrounded sum of 100%, this further part will produce a fractional rounding error of the same magnitude but opposite in sign to fractional rounding error of the sum of the first $n-1$ rounded parts, and so we can take that it as having the same distribution as each of the others. Since this rounding error is less in magnitude than $\frac{1}{2}$ of the rounding precision, it does not add anything to the visible rounding difference caused by the first $n-1$ parts, and this means that if the signed difference between the sum of the n rounded parts and 100% was d times the rounding precision and the probability was $J_n(d)$ then

$$J_n(d) = H_{n-1}(d).$$

4 Using the central limit theorem

Tables for $F_n(x)$, $G_n(x)$ and $H_n(d)$, and so implicitly $J_n(d)$, are shown in later sections. But since these are based on the sum of independent identical distributions with finite variances, we can use the central limit theorem to use a Gaussian distribution to approximate these values.

The mean of one rounding error is 0 with a standard deviation of $\sqrt{\frac{1}{12}}$ times the rounding precision. So the sum of n parts also has mean 0 and standard deviation $\sqrt{\frac{n}{12}}$. Writing $\Phi(x)$ for the cumulative distribution function of a standard Gaussian random variable with mean 0 and standard deviation 1, and $\phi(x)$ as the density, we then have

$$F_n(x) \approx \Phi\left(\sqrt{\frac{12}{n}}x\right)$$

and thus

$$G_n(x) \approx 2\Phi\left(\sqrt{\frac{12}{n}}x\right) - 1$$

and

$$H_n(d) \approx \Phi\left(\sqrt{\frac{12}{n}}\left(d+\frac{1}{2}\right)\right) - \Phi\left(\sqrt{\frac{12}{n}}\left(d-\frac{1}{2}\right)\right) \approx \sqrt{\frac{12}{n}}\phi\left(\sqrt{\frac{12}{n}}d\right).$$

Using this Gaussian approximation for $F_n(x)$ produces values which have a difference of less than 0.001 from the correct figure for $|x| > 2$ or $n > 11$, but requires much larger values to achieve better levels of accuracy. Since the approximations for $G_n(x)$ are twice as inaccurate, to be sure of figures which are within 0.001 of the correct figure, these become $x > 3$ or $n > 56$. The approximation for $G_{100}(5)$ is within 10^{-7} of the correct figure, but this is fortuitous: for $G_{100}(2)$ the approximation is more than 5×10^{-4} too high, while for $G_{100}(7)$ the approximation is over 1.4×10^{-4} too low.

As magnitudes, the Gaussian approximations are dreadful in the tails: for example with $|d| \geq 39$ and $n \leq 100$ they give magnitudes for $H_n(d)$ more than 10^{11} times the true figure, but this is in terms of very small probabilities as the true figure for $H_{100}(39)$ is about 1.244×10^{-52} while the approximations suggest values of around 7×10^{-41}or 3×10^{-41}.

The second Gaussian approximation given for $H_n(d)$ is notably weaker for $H_n(0)$, the probability of no visible rounding error. Better than either appears to be something like

$$H_n(0) = G_n(\tfrac{1}{2}) \approx \sqrt{\frac{6}{\pi\,(n+1.3)}}$$

and so

$$J_n(0) \approx \sqrt{\frac{6}{\pi\,(n+0.3)}}.$$

5 Rounding a geometric series

It is easy to find cases of rounding large numbers of small parts. Take for example this geometric series

$$1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \frac{6561}{10000} + \cdots = 10.$$

With a rounding precision of 1, the rounded terms are just the sum of seven ones giving a sum of the rounded parts of 7, and so a rounding difference of -3. But taking a rounding precision of 0.01, the first 51 parts each round to a positive number, and their sum is 9.98, giving a rounding difference of -2 times the rounding precision. Going further, a rounding precision of 0.000001 requires the sum of 138 rounded parts making 10.000001 with a visible rounding difference of $+1$ times the rounding precision. So the rounding difference is variable; as it also depends on the factor in the geometric series, we may be able to approximate some kind of expected rounding difference.

If we have a series of the form

$$\sum_{j=1}^{\infty} bk^j$$

for some constants $b > 0$ and $0 < k < 1$, and use a rounding precision of $p > 0$ with b substantially greater than p, then the terms which round to exactly p lie in the interval $\left[\frac{1}{2}p, \frac{3}{2}p\right]$. We can expect about $\frac{\log\left(\frac{3p}{2}\right)-\log\left(\frac{p}{2}\right)}{-\log(k)} = \frac{\log(3)}{-\log(k)}$ of them, and similarly about $\frac{\log(2i+1)-\log(2i-1)}{-\log(k)}$ terms rounding to ip, and so on all the way up to b; positive rounding errors are more likely than negative ones. We can use Stirling's formula to find that the expected total rounding error associated with terms rounding to a positive number is about

$$\sum_{i=1}^{\frac{b}{p}} \int_{i-\frac{1}{2}}^{\min(i+\frac{1}{2},\frac{b}{p})} \frac{i-x}{-x\log_e(k)}dx \approx \frac{\frac{b}{p}\log_e\left(\frac{b}{p}\right) - \left(\frac{b}{p}-\frac{1}{2}\right) - \sum_{i=1}^{\frac{b}{p}}\log_e\left(i-\frac{1}{2}\right)}{-\log_e(k)}$$

$$= \frac{\frac{1}{2} + \log_e\left(\left(\frac{4}{e}\frac{b}{p}\right)^{\frac{b}{p}} \frac{\left(\frac{b}{p}\right)!}{\left(\frac{2b}{p}\right)!}\right)}{-\log_e(k)} \approx \frac{1-\log_e(2)}{-2\log_e(k)}$$

times the rounding precision.

For the terms which round to zero, the largest term lies in the interval $\left[\frac{k}{2}p, \frac{1}{2}p\right]$ and so the negative of their sum is the rounding error and lies between $\frac{-k}{2(1-k)}$ and $\frac{-1}{2(1-k)}$ times the rounding precision. Using a similar assumption for the distribution to that implicitly assumed for the terms rounding to positive numbers (not that it makes much difference at this stage except for neatness), a reasonable central figure to take is $\frac{1}{2\log_e(k)}$ times the rounding precision. This is negative and larger than the approximate expectation for the rounding error for terms rounding to positive numbers.

So the overall expectation for the rounding error for the sum of a geometric series is about

$$\frac{\log(2)}{2\log(k)}$$

times the rounding precision, and is negative since $k < 1$; given its relatively small size and the variability of rounding errors, many actual rounded geometric series can have a positive rounding error. For $k = \frac{9}{10}$ shown in the example at the start of this section, it suggests a figure of roughly -3.3 times the rounding precision. With tighter precision the expected rounding error tends to get smaller in absolute terms, but its expected value does not change substantially as a multiple of the rounding precision.

6 Rounding a power-law series

The previous example showed a case where the expected rounding error was small. It is possible to make it much larger: take for example this power-law series, the Basel problem calculating $\zeta(2)$:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6} \approx 1.644934.$$

With a rounding precision of 1, the rounded terms are just $1+0+0+0+0+\cdots$ giving a sum of the rounded parts of 1, while rounding the actual sum gives 2. But taking a rounding precision of 0.01, the first fourteen parts round to a positive figure, and their sum is 1.59 while rounding the actual sum gives 1.64 giving a rounding difference of -5 times the rounding precision. Going further, a rounding precision of 0.000001 requires the sum of 1414 rounded parts to give 1.644332 giving a visible rounding difference of -602 times the rounding precision.

So making the rounding precision smaller tends to make the rounding error and the visible rounding difference a more negative multiple of the rounding precision. The exponent of the power law series also has an impact and here we used -2: it is no surprise that the rounding error gets worse more quickly if the exponent is -1.5 than if it is -4 as we have more parts close to the rounding precision. If we have a series of the form

$$\sum_{j=1}^{\infty} bj^{-k}$$

for some constants $b > 0$ and $k > 1$, and use a rounding precision of $p > 0$, then some experimentation suggests that the expected rounding error and visible rounding difference for a rounded power-law series might be something of the order of

$$-0.6(k-1)^{-1.2}\left(\frac{b}{p}\right)^{\frac{1}{k}}$$

times the rounding precision. It will not be that precise value both because the experimentation did not deliver a precise result and because of the natural distribution of errors, but for small k and large $\frac{b}{p}$ it will dominate. Taking $b = 1$ and $k = 2$ as in the example at the beginning of this section, it gives expected rounding errors of -0.6 when the rounding precision is 1; -6 times the rounding precision when that is 0.01; and -600 times the rounding precision when that is 0.000001. With tighter precision the expected rounding error tends to get smaller in absolute terms, but tends to get bigger in magnitude as a multiple of the rounding precision.

7 A realistic distribution for percentages

The earlier section on percentages suggested it is possible to have identical locally uniform distributions for the parts of 100%, that is to say of 1. But at a global level, these is less realistic. It might be more plausible to consider a distribution which has a constant density subject to the constraints that each of the n parts is non-negative and their sum is 1. The density turns out to be $(n-1)!$. It produces the following conditional distributions where k parts are known: if we label the known parts in any order A_i with values a_i for $1 \le i \le k$, and the unknown parts in any order A_{k+j} for $1 \le j \le n-k-1$, we have

$$\text{Prob}(A_{k+j} \leq x | A_1 = a_1, \ldots, A_k = a_k) = 1 - \left(1 - \frac{x}{1 - \sum_{i=1}^{k} a_i}\right)^{n-k-1}$$

$$\text{when } 0 \leq x \leq 1 - \sum_{i=1}^{k} a_i$$

and finally when $n-1$ parts are known

$$A_n = 1 - \sum_{i=1}^{n-1} a_i$$

so each part is identically distributed. Even if the overall density is constant, each part is more likely to be smaller than larger, as its marginal probability density is

$$f(x) = (n-1)(1-x)^{n-2} \text{ for } 0 \leq x \leq 1$$

which is decreasing in x so the signed rounding error as the difference between the rounded and the unrounded value of each part is therefore more likely to be negative than positive; the expected value of each part is of course $\frac{1}{n}$.

Imagine there are just three parts and the rounding precision is 50%, so each part is rounded to 0%, 50% or 100%. This is unrealistic, but is designed to demonstrate the idea simply enough to fit on the page while conveying the calculation. So a part will produce negative rounding error if its unrounded part is between 0% and 25% or between 50% or 75%, and positive rounding error otherwise. But there may be offsetting rounding errors from the other parts. To get a negative visible rounding difference from the three rounded parts, two must round to 0% and one to 50% adding to 50%, while to get a positive visible rounding difference each must round to 50% adding to 150%; with more parts or a smaller rounding precision there would be many more possibilities. The probability of a negative visible rounding difference is

$$\int_0^{\frac{1}{4}} \int_{\frac{1}{4}-x}^{\frac{1}{4}} 2\,dy\,dx + \int_0^{\frac{1}{4}} \int_{\frac{3}{4}-x}^{\frac{3}{4}} 2\,dy\,dx + \int_{\frac{1}{2}}^{\frac{3}{4}} \int_{\frac{3}{4}-x}^{\frac{1}{4}} 2\,dy\,dx = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}$$

while the probability of a positive visible rounding difference is

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{4}}^{\frac{3}{4}-x} 2\,dy\,dx = \frac{1}{16}.$$

The earlier uniform approximation gave $J_3(-1) = J_3(1) = \frac{1}{8}$. With this more realistic distribution, we find that with rounded percentages there is a higher probability of the sum of the rounded parts being less than 100%, as expected.

That example used a very large rounding precision. Using a more obvious rounding precision of 1% would give probabilities of negative and positive visible rounding differences of $\frac{101}{800}$ and $\frac{99}{800}$ respectively; using 0.1% would give $\frac{1001}{8000}$ and $\frac{999}{8000}$ respectively, with an obvious pattern. This justifies the earlier approximation as having been reasonable, but only when the rounding precision is small compared with the average size of the parts.

With a rounding precision of p, we can work out the expected rounding error for each part and multiply by n to find the bias or expected total rounding error or visible rounding difference to be

$$\frac{n}{p}\sum_{j=0}^{\frac{1}{p}}\int_{\max(0,jp-\frac{1}{2})}^{\min(1,jp+\frac{1}{2})}(jp-x)(n-1)(1-x)^{n-2}dx = -\frac{1}{p}+n\left(\frac{p}{2}\right)^{n-1}\sum_{j=1}^{\frac{1}{p}}(2j-1)^{n-1}$$

times the rounding precision. This expected rounding error will be negative: for small np with relatively few parts compared with the precision, this will be minor and close to $-\frac{n(n-1)p}{24}$ times the rounding precision; for large np it will tend towards $-\frac{1}{p} + n(1-\frac{p}{2})^{n-1}$ times the rounding precision, as almost all the parts will round to 0 and so the sum of the rounded parts will be close to 0. For example, with 100 parts and a rounding precision of 0.1% the expected rounding error is about -0.41 times the rounding precision so we expect the sum of the rounded parts to be close to 100.0%, but then allowing for the dispersion of $J_{100}(d)$; with 1000 parts and a rounding precision of 1% the expected rounding error is about -93.3 times the rounding precision so we expect the sum of the rounded parts to be close to a hopeless 7%, again with some dispersion. With a fixed number of parts, tighter precision tends to reduce the magnitude of expected rounding error as a multiple of the rounding precision.

8 An real example: reported vote shares

Two different methods have been put forward for estimating visible rounding differences for percentages by making assumptions about plausible distributions for rounding errors. In reality, distributions will not exactly follow either of these assumptions, and this can affect the results though possibly not by enough to notice. This example illustrates such a case.

The Electoral Commission[2][3] published the results of the United Kingdom 2001 and 2005 parliamentary general elections showing the votes won by each candidate in each constituency. It also published the share of each candidate's vote in their constituency rounded to 0.1%, and in some constituencies the sums of these showed visible percentage differences. For example, in Aberavon in 2001, the votes for the seven candidates were 19063, 2955, 2933, 2296, 1960, 727 and 256, making a total of 30190 so the vote shares were shown as 63.1%, 9.8%, 9.7%, 7.6%, 6.5%, 2.4% and 0.8%, summing to 99.9% giving a visible difference of −0.1%.

In 2001 there were 3319 candidates in 659 constituencies, implying a mean number of about 5.04 candidates per constituency. The number in individual constituencies ranged from three to nine: 45 constituencies had three candidates, 198 had four, 208 had five, 134 had six, 49 had seven, 20 had eight, and 5 had nine. 1177 of the candidates lost their £500 deposits by receiving less than a 5% share of the valid votes in their constituency.

The election four years later in 2005 saw more candidates: there were 3554 candidates in 646 constituencies after some boundaries in Scotland had been redrawn, implying a mean number of about 5.50 candidates per constituency. The number in individual constituencies ranged from three to fifteen: 21 constituencies had three candidates, 136 had four, 215 had five, 128 had six, 92 had seven, 34 had eight, 17 had nine, 2 had ten, and 1 (Sedgefield) had an astonishing fifteen. 1385 of the candidates lost their deposits.

If we used the symmetric numbers for $J_n(d)$, weighted for the number of candidates per constituency, we could predict the number of constituencies with particular visible rounding differences. This table compares the actual distribution of sums of rounded vote percentages with these predictions.

Sum of rounded percentages	Actual in 2001	Predicted for 2001	Actual in 2005	Predicted for 2005
99.7%	0	0.01	0	0.03
99.8%	3	3.06	8	4.58
99.9%	144	125.97	117	129.48
100.0%	398	400.93	376	377.82
100.1%	111	125.97	138	129.48
100.2%	3	3.06	7	4.58
100.3%	0	0.01	0	0.03

χ^2-tests with four degrees of freedom would not suggest the actual figures were significantly different from the predictions.

The distribution of vote shares of candidates was not consistent with the constant density percentage distribution assumed earlier: that would have predicted about 649.3 and 766.1 lost deposits in 2001 and 2005 respectively, just over half of the actual values which reflected the large number of fringe candidates. Despite that, the number of candidates who had such a low share of the vote that it rounded to 0.0% was smaller than would have been predicted: the constant density distribution would have predicted about 7.2 and 8.6 while the actual figures in the two elections were 0 and 4. So the expected negative bias in the visible rounding differences might be overstated by the constant density percentage distribution, and it was in any case small since the predictions add up to about a net −0.06% and −0.07% across the all constituencies in each of the two elections; in fact the total net visible rounding differences were −3.3% and +1.9% in 2001 and 2005 respectively, with the natural dispersion of results overwhelming any bias.

Finally, the results also showed the majority in each constituency: the difference in votes between the most and second most popular candidates and similarly as a proportion of the total valid votes. As a vote share, this too showed visible rounding differences, and although it involves a subtraction rather than an addition, a similar analysis should be possible, leading to a distribution similar to $H_2(d)$ of $\frac{1}{8}$, $\frac{3}{4}$ and $\frac{1}{8}$; we use $H_n(d)$ rather than $J_n(d)$ since the two percentages are not constrained to come to a particular figure. Again taking Aberavon in 2001 as an example, the top two vote shares were given as 63.1% and 9.8% but the majority of 16108 votes was shown as 53.4%, giving a visible difference of −0.1%. The following table compares the actual and predicted visible rounding differences in the majorities.

Visible differences in majorities	Actual in 2001	Predicted for 2001	Actual in 2005	Predicted for 2005
−0.1%	90	82.375	93	80.75
0.0%	480	494.25	478	484.5
+0.1%	89	82.375	75	80.75

Again χ^2-tests, this time with two degrees of freedom, would not suggest the actual figures were significantly different from the predictions. If a Gaussian approximation had been used instead, they would seem significant.

9 Table of $F_n(x)$

The cumulative distribution $F_n(x)$: the probability that the signed rounding error of the sum of n rounded terms is less than or equal to x times the rounding precision. By symmetry $F_n(0) = \frac{1}{2}$ and $F_n(-x) = 1 - F_n(x)$.

n	$F_n(0.5)$	$F_n(1)$	$F_n(1.5)$	$F_n(2)$	$F_n(2.5)$	$F_n(3)$	$F_n(3.5)$	$F_n(4)$	$F_n(4.5)$	$F_n(5)$	$F_n(5.5)$
1	1										
2	0.875	1									
3	0.83333	0.97917	1								
4	0.79948	0.95833	0.99740	1							
5	0.775	0.93802	0.99167	0.99974	1						
6	0.75551	0.91944	0.98431	0.99861	0.99998	1					
7	0.73968	0.90260	0.97599	0.99662	0.99980	1.00000	1				
8	0.72646	0.88738	0.96724	0.99385	0.99937	0.99998	1.00000	1			
9	0.71521	0.87360	0.95836	0.99044	0.99861	0.99989	1.00000	1.00000	1		
10	0.70548	0.86110	0.94955	0.98654	0.99753	0.99972	0.99998	1.00000	1.00000	1	
11	0.69696	0.84970	0.94092	0.98226	0.99613	0.99943	0.99995	1.00000	1.00000	1.00000	1
12	0.68942	0.83927	0.93255	0.97772	0.99442	0.99899	0.99988	0.99999	1.00000	1.00000	1.00000
13	0.68269	0.82969	0.92447	0.97300	0.99245	0.99841	0.99976	0.99998	1.00000	1.00000	1.00000
14	0.67662	0.82085	0.91670	0.96817	0.99024	0.99767	0.99958	0.99995	1.00000	1.00000	1.00000
15	0.67112	0.81267	0.90924	0.96328	0.98784	0.99678	0.99934	0.99990	0.99999	1.00000	1.00000
16	0.66610	0.80506	0.90209	0.95837	0.98527	0.99575	0.99902	0.99983	0.99998	1.00000	1.00000
17	0.66150	0.79798	0.89524	0.95346	0.98255	0.99457	0.99863	0.99972	0.99996	1.00000	1.00000
18	0.65727	0.79136	0.88868	0.94860	0.97973	0.99327	0.99815	0.99959	0.99993	0.99999	1.00000
19	0.65335	0.78515	0.88239	0.94379	0.97681	0.99185	0.99759	0.99941	0.99988	0.99998	1.00000
20	0.64971	0.77932	0.87637	0.93904	0.97383	0.99032	0.99695	0.99920	0.99983	0.99997	1.00000
21	0.64631	0.77383	0.87059	0.93438	0.97079	0.98869	0.99624	0.99894	0.99975	0.99995	0.99999
22	0.64314	0.76865	0.86505	0.92980	0.96771	0.98698	0.99545	0.99863	0.99965	0.99993	0.99999
23	0.64016	0.76374	0.85973	0.92531	0.96461	0.98519	0.99458	0.99828	0.99953	0.99989	0.99998
24	0.63737	0.75910	0.85462	0.92091	0.96149	0.98334	0.99365	0.99788	0.99939	0.99985	0.99997
25	0.63473	0.75468	0.84971	0.91660	0.95837	0.98143	0.99265	0.99744	0.99922	0.99979	0.99995
26	0.63224	0.75049	0.84499	0.91240	0.95525	0.97946	0.99159	0.99694	0.99902	0.99973	0.99993
27	0.62988	0.74649	0.84044	0.90828	0.95214	0.97745	0.99047	0.99641	0.99880	0.99965	0.99991
28	0.62765	0.74268	0.83606	0.90427	0.94905	0.97541	0.98930	0.99582	0.99855	0.99955	0.99988
29	0.62552	0.73904	0.83183	0.90034	0.94597	0.97334	0.98808	0.99520	0.99827	0.99944	0.99984
30	0.62350	0.73555	0.82776	0.89651	0.94292	0.97124	0.98682	0.99453	0.99796	0.99932	0.99980
31	0.62158	0.73222	0.82382	0.89277	0.93990	0.96912	0.98552	0.99383	0.99762	0.99917	0.99974
32	0.61974	0.72902	0.82002	0.88912	0.93691	0.96698	0.98417	0.99308	0.99725	0.99901	0.99968
33	0.61798	0.72595	0.81634	0.88555	0.93395	0.96484	0.98280	0.99230	0.99686	0.99883	0.99961
34	0.61630	0.72300	0.81279	0.88206	0.93102	0.96269	0.98140	0.99148	0.99643	0.99864	0.99953
35	0.61468	0.72016	0.80935	0.87866	0.92813	0.96053	0.97996	0.99063	0.99598	0.99842	0.99944
36	0.61314	0.71742	0.80601	0.87534	0.92528	0.95837	0.97851	0.98975	0.99550	0.99819	0.99933
37	0.61165	0.71479	0.80278	0.87209	0.92246	0.95621	0.97703	0.98884	0.99500	0.99793	0.99922
38	0.61022	0.71225	0.79965	0.86892	0.91969	0.95405	0.97553	0.98790	0.99446	0.99766	0.99909
39	0.60884	0.70979	0.79661	0.86582	0.91695	0.95190	0.97402	0.98694	0.99391	0.99737	0.99895
40	0.60752	0.70742	0.79366	0.86279	0.91424	0.94976	0.97249	0.98595	0.99333	0.99706	0.99880
41	0.60624	0.70513	0.79079	0.85983	0.91158	0.94763	0.97095	0.98494	0.99273	0.99673	0.99864
42	0.60501	0.70291	0.78801	0.85694	0.90896	0.94550	0.96939	0.98391	0.99210	0.99639	0.99846
43	0.60381	0.70076	0.78530	0.85411	0.90637	0.94339	0.96783	0.98286	0.99145	0.99602	0.99828
44	0.60266	0.69868	0.78266	0.85134	0.90382	0.94129	0.96627	0.98179	0.99079	0.99564	0.99807
45	0.60155	0.69666	0.78010	0.84863	0.90131	0.93921	0.96469	0.98071	0.99010	0.99524	0.99786
46	0.60047	0.69470	0.77760	0.84597	0.89884	0.93714	0.96311	0.97961	0.98939	0.99482	0.99763
47	0.59942	0.69280	0.77517	0.84338	0.89640	0.93508	0.96153	0.97849	0.98867	0.99439	0.99739
48	0.59841	0.69096	0.77280	0.84084	0.89401	0.93304	0.95995	0.97736	0.98793	0.99394	0.99714
49	0.59743	0.68916	0.77049	0.83835	0.89164	0.93102	0.95837	0.97623	0.98717	0.99348	0.99688
50	0.59647	0.68742	0.76823	0.83591	0.88932	0.92901	0.95678	0.97508	0.98640	0.99300	0.99660

$F_n(x)$ continued: the probability that the signed rounding error of the sum of n rounded terms is less than or equal to x times the rounding precision. $F_n(0) = \frac{1}{2}$ and $F_n(-x) = 1 - F_n(x)$.

n	$F_n(0.5)$	$F_n(1)$	$F_n(1.5)$	$F_n(2)$	$F_n(2.5)$	$F_n(3)$	$F_n(3.5)$	$F_n(4)$	$F_n(4.5)$	$F_n(5)$	$F_n(5.5)$
51	0.59555	0.68572	0.76603	0.83352	0.88702	0.92702	0.95520	0.97392	0.98562	0.99250	0.99631
52	0.59465	0.68407	0.76388	0.83118	0.88477	0.92505	0.95362	0.97275	0.98482	0.99199	0.99601
53	0.59377	0.68246	0.76179	0.82889	0.88254	0.92309	0.95205	0.97157	0.98400	0.99147	0.99569
54	0.59292	0.68089	0.75974	0.82664	0.88035	0.92116	0.95048	0.97039	0.98318	0.99093	0.99536
55	0.59209	0.67936	0.75774	0.82443	0.87819	0.91924	0.94891	0.96921	0.98234	0.99038	0.99503
56	0.59128	0.67787	0.75578	0.82227	0.87607	0.91734	0.94734	0.96801	0.98149	0.98982	0.99468
57	0.59050	0.67642	0.75387	0.82015	0.87397	0.91546	0.94579	0.96682	0.98064	0.98924	0.99432
58	0.58973	0.67500	0.75200	0.81806	0.87191	0.91359	0.94424	0.96561	0.97977	0.98866	0.99394
59	0.58898	0.67361	0.75017	0.81602	0.86988	0.91175	0.94269	0.96441	0.97889	0.98806	0.99356
60	0.58825	0.67226	0.74838	0.81401	0.86787	0.90992	0.94115	0.96321	0.97801	0.98745	0.99317
61	0.58754	0.67094	0.74662	0.81204	0.86590	0.90812	0.93962	0.96200	0.97712	0.98683	0.99277
62	0.58685	0.66965	0.74491	0.81011	0.86395	0.90633	0.93810	0.96079	0.97622	0.98620	0.99235
63	0.58617	0.66839	0.74322	0.80821	0.86204	0.90456	0.93658	0.95958	0.97531	0.98557	0.99193
64	0.58551	0.66715	0.74157	0.80634	0.86015	0.90280	0.93508	0.95837	0.97440	0.98492	0.99150
65	0.58486	0.66594	0.73996	0.80451	0.85828	0.90107	0.93358	0.95716	0.97348	0.98426	0.99106
66	0.58423	0.66476	0.73838	0.80271	0.85645	0.89935	0.93209	0.95595	0.97256	0.98360	0.99061
67	0.58361	0.66360	0.73682	0.80094	0.85464	0.89765	0.93061	0.95474	0.97163	0.98293	0.99015
68	0.58300	0.66247	0.73530	0.79919	0.85285	0.89597	0.92914	0.95353	0.97070	0.98225	0.98968
69	0.58241	0.66136	0.73380	0.79748	0.85109	0.89431	0.92767	0.95233	0.96976	0.98157	0.98920
70	0.58183	0.66027	0.73234	0.79580	0.84935	0.89266	0.92622	0.95112	0.96883	0.98087	0.98872
71	0.58126	0.65920	0.73090	0.79414	0.84764	0.89103	0.92478	0.94992	0.96788	0.98017	0.98823
72	0.58071	0.65816	0.72948	0.79251	0.84595	0.88942	0.92334	0.94872	0.96694	0.97947	0.98773
73	0.58016	0.65713	0.72810	0.79091	0.84429	0.88783	0.92192	0.94753	0.96599	0.97876	0.98723
74	0.57963	0.65613	0.72673	0.78933	0.84264	0.88625	0.92050	0.94634	0.96504	0.97804	0.98671
75	0.57911	0.65514	0.72539	0.78778	0.84102	0.88469	0.91910	0.94515	0.96409	0.97732	0.98619
76	0.57859	0.65417	0.72408	0.78625	0.83942	0.88314	0.91770	0.94396	0.96314	0.97660	0.98567
77	0.57809	0.65322	0.72279	0.78475	0.83784	0.88161	0.91632	0.94278	0.96219	0.97587	0.98514
78	0.57759	0.65229	0.72151	0.78326	0.83628	0.88010	0.91494	0.94160	0.96123	0.97513	0.98460
79	0.57711	0.65138	0.72026	0.78180	0.83474	0.87860	0.91358	0.94043	0.96028	0.97439	0.98405
80	0.57663	0.65048	0.71904	0.78037	0.83323	0.87712	0.91222	0.93926	0.95932	0.97365	0.98351
81	0.57617	0.64959	0.71783	0.77895	0.83173	0.87565	0.91087	0.93810	0.95837	0.97291	0.98295
82	0.57571	0.64872	0.71664	0.77756	0.83025	0.87420	0.90954	0.93694	0.95741	0.97216	0.98239
83	0.57526	0.64787	0.71547	0.77618	0.82879	0.87276	0.90821	0.93578	0.95646	0.97141	0.98183
84	0.57482	0.64703	0.71432	0.77483	0.82734	0.87134	0.90690	0.93463	0.95550	0.97065	0.98126
85	0.57438	0.64621	0.71318	0.77349	0.82592	0.86993	0.90559	0.93349	0.95455	0.96990	0.98068
86	0.57395	0.64540	0.71207	0.77217	0.82451	0.86853	0.90429	0.93235	0.95360	0.96914	0.98011
87	0.57353	0.64460	0.71097	0.77088	0.82312	0.86715	0.90301	0.93121	0.95264	0.96838	0.97953
88	0.57312	0.64381	0.70989	0.76960	0.82175	0.86579	0.90173	0.93008	0.95169	0.96761	0.97894
89	0.57272	0.64304	0.70883	0.76833	0.82039	0.86444	0.90046	0.92896	0.95075	0.96685	0.97835
90	0.57232	0.64228	0.70778	0.76709	0.81905	0.86310	0.89921	0.92784	0.94980	0.96608	0.97776
91	0.57192	0.64153	0.70674	0.76586	0.81773	0.86177	0.89796	0.92673	0.94885	0.96531	0.97716
92	0.57154	0.64080	0.70572	0.76465	0.81642	0.86046	0.89672	0.92562	0.94791	0.96454	0.97656
93	0.57116	0.64007	0.70472	0.76345	0.81513	0.85916	0.89549	0.92452	0.94696	0.96377	0.97596
94	0.57078	0.63936	0.70373	0.76228	0.81385	0.85788	0.89427	0.92342	0.94602	0.96300	0.97535
95	0.57041	0.63866	0.70276	0.76111	0.81259	0.85660	0.89306	0.92233	0.94509	0.96223	0.97474
96	0.57005	0.63796	0.70180	0.75996	0.81134	0.85534	0.89186	0.92124	0.94415	0.96146	0.97413
97	0.56969	0.63728	0.70085	0.75883	0.81011	0.85409	0.89067	0.92016	0.94322	0.96069	0.97352
98	0.56934	0.63661	0.69992	0.75771	0.80889	0.85286	0.88949	0.91909	0.94229	0.95991	0.97290
99	0.56900	0.63595	0.69900	0.75661	0.80769	0.85163	0.88831	0.91802	0.94136	0.95914	0.97229
100	0.56865	0.63530	0.69809	0.75551	0.80649	0.85042	0.88715	0.91696	0.94043	0.95837	0.97167

10 Table of $G_n(x)$

$G_n(x)$: the probability that the absolute rounding error of the sum of n rounded terms is less than or equal to x times the rounding precision. We have $G_n(0) = 0$ since $G_n(x) = 2F_n(x) - 1$.

n	$G_n(0.5)$	$G_n(1)$	$G_n(1.5)$	$G_n(2)$	$G_n(2.5)$	$G_n(3)$	$G_n(3.5)$	$G_n(4)$	$G_n(4.5)$	$G_n(5)$	$G_n(5.5)$
1	1										
2	0.75	1									
3	0.66667	0.95833	1								
4	0.59896	0.91667	0.99479	1							
5	0.55	0.87604	0.98333	0.99948	1						
6	0.51102	0.83889	0.96862	0.99722	0.99996	1					
7	0.47937	0.80519	0.95198	0.99324	0.99960	1.00000	1				
8	0.45292	0.77475	0.93448	0.98770	0.99873	0.99995	1.00000	1			
9	0.43042	0.74720	0.91672	0.98088	0.99723	0.99979	0.99999	1.00000	1		
10	0.41096	0.72220	0.89909	0.97307	0.99506	0.99944	0.99997	1.00000	1.00000	1	
11	0.39393	0.69940	0.88185	0.96453	0.99225	0.99885	0.99990	1.00000	1.00000	1.00000	1
12	0.37884	0.67855	0.86511	0.95545	0.98884	0.99799	0.99976	0.99998	1.00000	1.00000	1.00000
13	0.36537	0.65938	0.84895	0.94601	0.98490	0.99682	0.99952	0.99995	1.00000	1.00000	1.00000
14	0.35324	0.64170	0.83341	0.93634	0.98049	0.99534	0.99917	0.99990	0.99999	1.00000	1.00000
15	0.34224	0.62533	0.81849	0.92656	0.97568	0.99356	0.99868	0.99980	0.99998	1.00000	1.00000
16	0.33221	0.61013	0.80418	0.91673	0.97053	0.99149	0.99805	0.99965	0.99996	1.00000	1.00000
17	0.32301	0.59596	0.79048	0.90693	0.96511	0.98914	0.99725	0.99945	0.99992	0.99999	1.00000
18	0.31453	0.58272	0.77736	0.89720	0.95945	0.98654	0.99630	0.99918	0.99986	0.99998	1.00000
19	0.30669	0.57031	0.76478	0.88757	0.95362	0.98370	0.99519	0.99883	0.99977	0.99996	1.00000
20	0.29941	0.55864	0.75274	0.87809	0.94765	0.98064	0.99391	0.99839	0.99965	0.99994	0.99999
21	0.29262	0.54766	0.74118	0.86876	0.94157	0.97739	0.99248	0.99787	0.99950	0.99990	0.99998
22	0.28628	0.53729	0.73010	0.85959	0.93542	0.97396	0.99089	0.99726	0.99930	0.99985	0.99997
23	0.28033	0.52749	0.71946	0.85061	0.92921	0.97039	0.98916	0.99656	0.99906	0.99978	0.99996
24	0.27473	0.51819	0.70924	0.84182	0.92298	0.96668	0.98729	0.99576	0.99878	0.99970	0.99994
25	0.26946	0.50937	0.69942	0.83321	0.91673	0.96285	0.98529	0.99487	0.99844	0.99959	0.99991
26	0.26448	0.50098	0.68997	0.82479	0.91050	0.95892	0.98317	0.99389	0.99805	0.99946	0.99987
27	0.25977	0.49298	0.68088	0.81657	0.90428	0.95491	0.98094	0.99281	0.99760	0.99930	0.99982
28	0.25530	0.48536	0.67211	0.80853	0.89809	0.95082	0.97860	0.99165	0.99710	0.99911	0.99976
29	0.25105	0.47808	0.66366	0.80069	0.89194	0.94667	0.97616	0.99040	0.99653	0.99889	0.99968
30	0.24701	0.47111	0.65551	0.79302	0.88585	0.94248	0.97364	0.98907	0.99592	0.99863	0.99959
31	0.24315	0.46444	0.64764	0.78554	0.87980	0.93824	0.97103	0.98765	0.99524	0.99835	0.99949
32	0.23948	0.45804	0.64004	0.77823	0.87382	0.93397	0.96835	0.98616	0.99450	0.99803	0.99936
33	0.23596	0.45190	0.63269	0.77110	0.86790	0.92968	0.96560	0.98460	0.99371	0.99767	0.99922
34	0.23259	0.44599	0.62558	0.76413	0.86205	0.92537	0.96279	0.98296	0.99286	0.99728	0.99906
35	0.22937	0.44031	0.61870	0.75733	0.85627	0.92105	0.95993	0.98126	0.99196	0.99684	0.99887
36	0.22627	0.43485	0.61203	0.75068	0.85056	0.91673	0.95701	0.97950	0.99100	0.99638	0.99867
37	0.22330	0.42958	0.60557	0.74419	0.84493	0.91242	0.95406	0.97768	0.98999	0.99587	0.99844
38	0.22044	0.42449	0.59930	0.73785	0.83937	0.90811	0.95106	0.97581	0.98893	0.99532	0.99818
39	0.21769	0.41959	0.59322	0.73165	0.83389	0.90381	0.94803	0.97388	0.98782	0.99474	0.99791
40	0.21504	0.41484	0.58732	0.72559	0.82849	0.89952	0.94497	0.97190	0.98666	0.99412	0.99761
41	0.21248	0.41026	0.58158	0.71967	0.82316	0.89525	0.94189	0.96988	0.98545	0.99347	0.99728
42	0.21001	0.40582	0.57601	0.71388	0.81791	0.89101	0.93879	0.96782	0.98420	0.99277	0.99693
43	0.20763	0.40152	0.57059	0.70821	0.81274	0.88678	0.93567	0.96572	0.98291	0.99204	0.99655
44	0.20533	0.39736	0.56532	0.70267	0.80764	0.88258	0.93253	0.96358	0.98157	0.99128	0.99615
45	0.20310	0.39332	0.56019	0.69725	0.80262	0.87841	0.92938	0.96141	0.98020	0.99048	0.99572
46	0.20094	0.38941	0.55520	0.69195	0.79768	0.87427	0.92623	0.95921	0.97879	0.98965	0.99527
47	0.19885	0.38561	0.55033	0.68676	0.79281	0.87016	0.92307	0.95698	0.97734	0.98878	0.99479
48	0.19682	0.38191	0.54559	0.68167	0.78801	0.86608	0.91990	0.95473	0.97586	0.98788	0.99428
49	0.19485	0.37832	0.54097	0.67670	0.78328	0.86203	0.91674	0.95245	0.97435	0.98695	0.99375
50	0.19295	0.37483	0.53646	0.67182	0.77863	0.85802	0.91357	0.95015	0.97281	0.98599	0.99320

$G_n(x)$ continued: the probability that the absolute rounding error of the sum of n rounded terms is less than or equal to x times the rounding precision. $G_n(0) = 0$.

n	$G_n(0.5)$	$G_n(1)$	$G_n(1.5)$	$G_n(2)$	$G_n(2.5)$	$G_n(3)$	$G_n(3.5)$	$G_n(4)$	$G_n(4.5)$	$G_n(5)$	$G_n(5.5)$
51	0.19109	0.37144	0.53206	0.66704	0.77405	0.85404	0.91041	0.94783	0.97123	0.98500	0.99261
52	0.18929	0.36813	0.52777	0.66236	0.76953	0.85009	0.90725	0.94550	0.96963	0.98398	0.99201
53	0.18754	0.36492	0.52357	0.65777	0.76508	0.84618	0.90410	0.94315	0.96801	0.98293	0.99138
54	0.18584	0.36178	0.51948	0.65327	0.76070	0.84231	0.90095	0.94079	0.96636	0.98186	0.99073
55	0.18418	0.35872	0.51548	0.64886	0.75639	0.83848	0.89782	0.93841	0.96468	0.98076	0.99005
56	0.18256	0.35574	0.51156	0.64454	0.75213	0.83468	0.89469	0.93603	0.96299	0.97963	0.98935
57	0.18099	0.35284	0.50774	0.64029	0.74795	0.83091	0.89157	0.93363	0.96127	0.97848	0.98863
58	0.17946	0.35000	0.50400	0.63613	0.74382	0.82719	0.88847	0.93123	0.95954	0.97731	0.98789
59	0.17797	0.34723	0.50034	0.63204	0.73975	0.82350	0.88538	0.92882	0.95778	0.97612	0.98712
60	0.17651	0.34452	0.49675	0.62803	0.73575	0.81985	0.88231	0.92641	0.95602	0.97490	0.98634
61	0.17509	0.34188	0.49325	0.62409	0.73180	0.81623	0.87924	0.92399	0.95423	0.97366	0.98553
62	0.17370	0.33930	0.48981	0.62022	0.72791	0.81266	0.87620	0.92158	0.95243	0.97241	0.98471
63	0.17234	0.33677	0.48645	0.61642	0.72407	0.80911	0.87317	0.91916	0.95062	0.97113	0.98386
64	0.17102	0.33430	0.48315	0.61269	0.72029	0.80561	0.87015	0.91674	0.94880	0.96984	0.98300
65	0.16972	0.33189	0.47992	0.60902	0.71657	0.80214	0.86716	0.91432	0.94696	0.96853	0.98211
66	0.16846	0.32952	0.47675	0.60541	0.71289	0.79871	0.86418	0.91190	0.94512	0.96720	0.98121
67	0.16722	0.32720	0.47364	0.60187	0.70927	0.79531	0.86121	0.90948	0.94326	0.96586	0.98029
68	0.16601	0.32494	0.47060	0.59839	0.70570	0.79195	0.85827	0.90707	0.94140	0.96450	0.97936
69	0.16482	0.32272	0.46761	0.59496	0.70218	0.78862	0.85535	0.90465	0.93953	0.96313	0.97841
70	0.16366	0.32054	0.46467	0.59160	0.69871	0.78533	0.85244	0.90225	0.93765	0.96175	0.97744
71	0.16253	0.31841	0.46180	0.58828	0.69528	0.78207	0.84955	0.89985	0.93577	0.96035	0.97646
72	0.16142	0.31632	0.45897	0.58502	0.69190	0.77884	0.84668	0.89745	0.93388	0.95894	0.97546
73	0.16033	0.31427	0.45619	0.58182	0.68857	0.77565	0.84383	0.89506	0.93198	0.95752	0.97445
74	0.15926	0.31226	0.45347	0.57866	0.68529	0.77250	0.84100	0.89267	0.93009	0.95609	0.97342
75	0.15821	0.31028	0.45079	0.57556	0.68204	0.76937	0.83819	0.89030	0.92819	0.95464	0.97239
76	0.15718	0.30835	0.44816	0.57250	0.67884	0.76628	0.83540	0.88793	0.92628	0.95319	0.97134
77	0.15618	0.30645	0.44557	0.56949	0.67568	0.76322	0.83263	0.88556	0.92437	0.95173	0.97027
78	0.15519	0.30458	0.44303	0.56653	0.67257	0.76019	0.82988	0.88321	0.92247	0.95026	0.96920
79	0.15422	0.30275	0.44053	0.56361	0.66949	0.75720	0.82715	0.88086	0.92056	0.94879	0.96811
80	0.15327	0.30095	0.43807	0.56073	0.66645	0.75423	0.82444	0.87852	0.91865	0.94730	0.96701
81	0.15233	0.29919	0.43565	0.55790	0.66346	0.75130	0.82175	0.87620	0.91674	0.94581	0.96590
82	0.15142	0.29745	0.43328	0.55511	0.66050	0.74839	0.81908	0.87388	0.91482	0.94432	0.96478
83	0.15052	0.29574	0.43094	0.55236	0.65757	0.74552	0.81643	0.87157	0.91292	0.94281	0.96365
84	0.14963	0.29406	0.42863	0.54965	0.65469	0.74267	0.81379	0.86927	0.91101	0.94130	0.96252
85	0.14876	0.29242	0.42637	0.54698	0.65184	0.73986	0.81118	0.86697	0.90910	0.93979	0.96137
86	0.14791	0.29079	0.42414	0.54435	0.64903	0.73707	0.80859	0.86469	0.90719	0.93827	0.96021
87	0.14707	0.28920	0.42194	0.54175	0.64625	0.73431	0.80602	0.86242	0.90529	0.93675	0.95905
88	0.14624	0.28763	0.41978	0.53919	0.64350	0.73158	0.80346	0.86017	0.90339	0.93522	0.95788
89	0.14543	0.28608	0.41765	0.53667	0.64079	0.72887	0.80093	0.85792	0.90149	0.93369	0.95670
90	0.14463	0.28456	0.41555	0.53418	0.63811	0.72620	0.79841	0.85568	0.89959	0.93216	0.95551
91	0.14385	0.28307	0.41349	0.53172	0.63546	0.72355	0.79592	0.85345	0.89770	0.93063	0.95432
92	0.14307	0.28160	0.41145	0.52930	0.63285	0.72092	0.79344	0.85124	0.89581	0.92909	0.95312
93	0.14231	0.28015	0.40944	0.52691	0.63026	0.71832	0.79098	0.84903	0.89393	0.92755	0.95192
94	0.14156	0.27872	0.40747	0.52455	0.62771	0.71575	0.78854	0.84684	0.89205	0.92601	0.95070
95	0.14083	0.27731	0.40552	0.52222	0.62518	0.71321	0.78612	0.84465	0.89017	0.92446	0.94949
96	0.14010	0.27593	0.40360	0.51993	0.62269	0.71068	0.78372	0.84248	0.88830	0.92292	0.94827
97	0.13939	0.27456	0.40170	0.51766	0.62022	0.70819	0.78134	0.84032	0.88643	0.92137	0.94704
98	0.13868	0.27322	0.39983	0.51542	0.61778	0.70571	0.77897	0.83818	0.88457	0.91983	0.94581
99	0.13799	0.27190	0.39799	0.51321	0.61537	0.70326	0.77663	0.83604	0.88271	0.91828	0.94457
100	0.13731	0.27059	0.39617	0.51103	0.61299	0.70084	0.77430	0.83391	0.88086	0.91674	0.94333

11 Table of $H_n(d)$

$H_n(d)$: the probability that the signed visible rounding difference between the sum of n rounded terms and the rounded sum is equal to d times the rounding precision. $H_n(d) = F_n(d+\frac{1}{2}) - F_n(d-\frac{1}{2})$, so by symmetry $H_n(-d) = H_n(d)$. If instead percentages are used making the sum 100%, then the probability is about $J_n(d) = H_{n-1}(d)$ times the rounding precision, so using the preceding row.

n	$H_n(0)$	$H_n(1)$	$H_n(2)$	$H_n(3)$	$H_n(4)$	$H_n(5)$	$H_n(6)$	$H_n(7)$	$H_n(8)$	$H_n(9)$	$H_n(10)$
1	1	0									
2	0.75	0.125									
3	0.66667	0.16667	0								
4	0.59896	0.19792	0.00260								
5	0.55	0.21667	0.00833	0							
6	0.51102	0.22880	0.01567	0.00002							
7	0.47937	0.23631	0.02381	0.00020	0						
8	0.45292	0.24078	0.03213	0.00063	0.00000						
9	0.43042	0.24315	0.04026	0.00138	0.00000	0					
10	0.41096	0.24407	0.04798	0.00245	0.00002	0.00000					
11	0.39393	0.24396	0.05520	0.00382	0.00005	0.00000	0				
12	0.37884	0.24313	0.06187	0.00546	0.00012	0.00000	0.00000				
13	0.36537	0.24179	0.06797	0.00731	0.00024	0.00000	0.00000	0			
14	0.35324	0.24008	0.07354	0.00934	0.00041	0.00000	0.00000	0.00000			
15	0.34224	0.23812	0.07860	0.01150	0.00065	0.00001	0.00000	0.00000	0		
16	0.33221	0.23599	0.08317	0.01376	0.00095	0.00002	0.00000	0.00000	0.00000		
17	0.32301	0.23374	0.08731	0.01607	0.00133	0.00004	0.00000	0.00000	0.00000	0	
18	0.31453	0.23141	0.09105	0.01842	0.00178	0.00007	0.00000	0.00000	0.00000	0.00000	
19	0.30669	0.22905	0.09442	0.02078	0.00229	0.00011	0.00000	0.00000	0.00000	0.00000	0
20	0.29941	0.22666	0.09746	0.02313	0.00287	0.00017	0.00000	0.00000	0.00000	0.00000	0.00000
21	0.29262	0.22428	0.10019	0.02545	0.00351	0.00024	0.00001	0.00000	0.00000	0.00000	0.00000
22	0.28628	0.22191	0.10266	0.02774	0.00421	0.00034	0.00001	0.00000	0.00000	0.00000	0.00000
23	0.28033	0.21957	0.10487	0.02997	0.00495	0.00045	0.00002	0.00000	0.00000	0.00000	0.00000
24	0.27473	0.21726	0.10687	0.03216	0.00574	0.00058	0.00003	0.00000	0.00000	0.00000	0.00000
25	0.26946	0.21498	0.10866	0.03428	0.00657	0.00073	0.00004	0.00000	0.00000	0.00000	0.00000
26	0.26448	0.21275	0.11026	0.03634	0.00744	0.00091	0.00006	0.00000	0.00000	0.00000	0.00000
27	0.25977	0.21055	0.11170	0.03833	0.00833	0.00111	0.00009	0.00000	0.00000	0.00000	0.00000
28	0.25530	0.20841	0.11299	0.04025	0.00925	0.00133	0.00012	0.00001	0.00000	0.00000	0.00000
29	0.25105	0.20631	0.11414	0.04211	0.01019	0.00157	0.00015	0.00001	0.00000	0.00000	0.00000
30	0.24701	0.20425	0.11517	0.04390	0.01114	0.00184	0.00019	0.00001	0.00000	0.00000	0.00000
31	0.24315	0.20224	0.11608	0.04561	0.01210	0.00212	0.00024	0.00002	0.00000	0.00000	0.00000
32	0.23948	0.20028	0.11689	0.04727	0.01308	0.00243	0.00030	0.00002	0.00000	0.00000	0.00000
33	0.23596	0.19837	0.11760	0.04885	0.01406	0.00275	0.00036	0.00003	0.00000	0.00000	0.00000
34	0.23259	0.19649	0.11823	0.05037	0.01504	0.00310	0.00043	0.00004	0.00000	0.00000	0.00000
35	0.22937	0.19466	0.11879	0.05183	0.01602	0.00346	0.00051	0.00005	0.00000	0.00000	0.00000
36	0.22627	0.19288	0.11927	0.05323	0.01699	0.00383	0.00060	0.00006	0.00000	0.00000	0.00000
37	0.22330	0.19113	0.11968	0.05456	0.01797	0.00422	0.00070	0.00008	0.00001	0.00000	0.00000
38	0.22044	0.18943	0.12003	0.05585	0.01893	0.00463	0.00080	0.00010	0.00001	0.00000	0.00000
39	0.21769	0.18777	0.12033	0.05707	0.01989	0.00504	0.00092	0.00012	0.00001	0.00000	0.00000
40	0.21504	0.18614	0.12058	0.05824	0.02084	0.00547	0.00104	0.00014	0.00001	0.00000	0.00000
41	0.21248	0.18455	0.12079	0.05937	0.02178	0.00591	0.00117	0.00017	0.00002	0.00000	0.00000
42	0.21001	0.18300	0.12095	0.06044	0.02271	0.00636	0.00132	0.00020	0.00002	0.00000	0.00000
43	0.20763	0.18148	0.12107	0.06146	0.02362	0.00682	0.00147	0.00023	0.00003	0.00000	0.00000
44	0.20533	0.18000	0.12116	0.06244	0.02452	0.00729	0.00162	0.00027	0.00003	0.00000	0.00000
45	0.20310	0.17855	0.12121	0.06338	0.02541	0.00776	0.00179	0.00031	0.00004	0.00000	0.00000
46	0.20094	0.17713	0.12124	0.06427	0.02628	0.00824	0.00196	0.00035	0.00005	0.00000	0.00000
47	0.19885	0.17574	0.12124	0.06513	0.02714	0.00872	0.00215	0.00040	0.00006	0.00001	0.00000
48	0.19682	0.17439	0.12121	0.06595	0.02798	0.00921	0.00233	0.00045	0.00007	0.00001	0.00000
49	0.19485	0.17306	0.12116	0.06673	0.02881	0.00970	0.00253	0.00051	0.00008	0.00001	0.00000
50	0.19295	0.17176	0.12108	0.06747	0.02962	0.01019	0.00273	0.00057	0.00009	0.00001	0.00000

$H_n(d)$ continued: the probability that the signed visible rounding difference between the sum of n rounded terms and the rounded sum is equal to d times the rounding precision. $H_n(-d) = H_n(d)$. For percentages, the probability is about $J_n(d) = H_{n-1}(d)$ times the rounding precision.

n	$H_n(0)$	$H_n(1)$	$H_n(2)$	$H_n(3)$	$H_n(4)$	$H_n(5)$	$H_n(6)$	$H_n(7)$	$H_n(8)$	$H_n(9)$	$H_n(10)$
51	0.19109	0.17049	0.12099	0.06818	0.03041	0.01069	0.00294	0.00063	0.00010	0.00001	0.00000
52	0.18929	0.16924	0.12088	0.06886	0.03119	0.01119	0.00316	0.00070	0.00012	0.00002	0.00000
53	0.18754	0.16802	0.12075	0.06951	0.03196	0.01169	0.00338	0.00077	0.00014	0.00002	0.00000
54	0.18584	0.16682	0.12061	0.07013	0.03270	0.01219	0.00361	0.00085	0.00016	0.00002	0.00000
55	0.18418	0.16565	0.12045	0.07071	0.03343	0.01269	0.00384	0.00092	0.00018	0.00003	0.00000
56	0.18256	0.16450	0.12028	0.07128	0.03415	0.01318	0.00408	0.00101	0.00020	0.00003	0.00000
57	0.18099	0.16337	0.12010	0.07181	0.03485	0.01368	0.00433	0.00110	0.00022	0.00004	0.00000
58	0.17946	0.16227	0.11991	0.07233	0.03553	0.01418	0.00457	0.00119	0.00025	0.00004	0.00001
59	0.17797	0.16119	0.11971	0.07281	0.03620	0.01467	0.00483	0.00128	0.00027	0.00005	0.00001
60	0.17651	0.16012	0.11950	0.07328	0.03686	0.01516	0.00508	0.00138	0.00030	0.00005	0.00001
61	0.17509	0.15908	0.11928	0.07372	0.03749	0.01565	0.00534	0.00149	0.00033	0.00006	0.00001
62	0.17370	0.15806	0.11905	0.07414	0.03812	0.01614	0.00561	0.00159	0.00037	0.00007	0.00001
63	0.17234	0.15705	0.11881	0.07455	0.03873	0.01662	0.00587	0.00170	0.00040	0.00008	0.00001
64	0.17102	0.15607	0.11857	0.07493	0.03932	0.01710	0.00614	0.00182	0.00044	0.00009	0.00001
65	0.16972	0.15510	0.11832	0.07529	0.03990	0.01758	0.00642	0.00193	0.00048	0.00010	0.00002
66	0.16846	0.15415	0.11807	0.07564	0.04047	0.01805	0.00669	0.00205	0.00052	0.00011	0.00002
67	0.16722	0.15321	0.11781	0.07597	0.04102	0.01852	0.00697	0.00218	0.00056	0.00012	0.00002
68	0.16601	0.15229	0.11755	0.07628	0.04156	0.01898	0.00725	0.00231	0.00061	0.00013	0.00002
69	0.16482	0.15139	0.11729	0.07658	0.04209	0.01944	0.00753	0.00244	0.00066	0.00015	0.00003
70	0.16366	0.15051	0.11702	0.07687	0.04261	0.01989	0.00781	0.00257	0.00071	0.00016	0.00003
71	0.16253	0.14963	0.11674	0.07713	0.04311	0.02035	0.00809	0.00270	0.00076	0.00018	0.00003
72	0.16142	0.14878	0.11647	0.07739	0.04360	0.02079	0.00838	0.00284	0.00081	0.00019	0.00004
73	0.16033	0.14793	0.11619	0.07763	0.04408	0.02123	0.00866	0.00298	0.00087	0.00021	0.00004
74	0.15926	0.14710	0.11591	0.07786	0.04454	0.02167	0.00895	0.00313	0.00092	0.00023	0.00005
75	0.15821	0.14629	0.11563	0.07808	0.04500	0.02210	0.00923	0.00328	0.00098	0.00025	0.00005
76	0.15718	0.14549	0.11534	0.07828	0.04544	0.02253	0.00952	0.00342	0.00104	0.00027	0.00006
77	0.15618	0.14470	0.11506	0.07848	0.04587	0.02295	0.00981	0.00358	0.00111	0.00029	0.00006
78	0.15519	0.14392	0.11477	0.07866	0.04629	0.02336	0.01010	0.00373	0.00117	0.00031	0.00007
79	0.15422	0.14315	0.11448	0.07883	0.04670	0.02378	0.01039	0.00388	0.00124	0.00034	0.00008
80	0.15327	0.14240	0.11419	0.07899	0.04710	0.02418	0.01067	0.00404	0.00131	0.00036	0.00009
81	0.15233	0.14166	0.11390	0.07915	0.04749	0.02458	0.01096	0.00420	0.00138	0.00039	0.00009
82	0.15142	0.14093	0.11361	0.07929	0.04787	0.02498	0.01125	0.00436	0.00145	0.00042	0.00010
83	0.15052	0.14021	0.11332	0.07943	0.04824	0.02537	0.01153	0.00453	0.00153	0.00044	0.00011
84	0.14963	0.13950	0.11303	0.07955	0.04861	0.02575	0.01182	0.00469	0.00161	0.00047	0.00012
85	0.14876	0.13880	0.11274	0.07967	0.04896	0.02614	0.01210	0.00486	0.00168	0.00050	0.00013
86	0.14791	0.13812	0.11244	0.07978	0.04930	0.02651	0.01239	0.00502	0.00176	0.00054	0.00014
87	0.14707	0.13744	0.11215	0.07989	0.04964	0.02688	0.01267	0.00519	0.00185	0.00057	0.00015
88	0.14624	0.13677	0.11186	0.07998	0.04996	0.02725	0.01295	0.00536	0.00193	0.00060	0.00016
89	0.14543	0.13611	0.11157	0.08007	0.05028	0.02760	0.01324	0.00553	0.00202	0.00064	0.00017
90	0.14463	0.13546	0.11128	0.08015	0.05059	0.02796	0.01352	0.00571	0.00210	0.00067	0.00019
91	0.14385	0.13482	0.11099	0.08023	0.05089	0.02831	0.01379	0.00588	0.00219	0.00071	0.00020
92	0.14307	0.13419	0.11070	0.08030	0.05119	0.02865	0.01407	0.00606	0.00228	0.00075	0.00021
93	0.14231	0.13357	0.11041	0.08036	0.05147	0.02899	0.01435	0.00623	0.00237	0.00079	0.00023
94	0.14156	0.13295	0.11012	0.08042	0.05175	0.02933	0.01462	0.00641	0.00246	0.00083	0.00024
95	0.14083	0.13235	0.10983	0.08047	0.05202	0.02966	0.01490	0.00658	0.00256	0.00087	0.00026
96	0.14010	0.13175	0.10955	0.08052	0.05229	0.02998	0.01517	0.00676	0.00265	0.00091	0.00028
97	0.13939	0.13116	0.10926	0.08056	0.05255	0.03030	0.01544	0.00694	0.00275	0.00096	0.00029
98	0.13868	0.13057	0.10897	0.08060	0.05280	0.03062	0.01571	0.00712	0.00285	0.00100	0.00031
99	0.13799	0.13000	0.10869	0.08063	0.05304	0.03093	0.01597	0.00730	0.00295	0.00105	0.00033
100	0.13731	0.12943	0.10841	0.08065	0.05328	0.03124	0.01624	0.00748	0.00305	0.00110	0.00035

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